Geometric sequences (answers at the end)

1 In a geometric progression, the fifth term is 100 and the seventh term is 400. Find the first term.

2 A geometric series has first term a and common ratio $\frac{1}{\sqrt{2}}$. Show that the sum to infinity of the series is $a(2+\sqrt{2})$.

3 The nth term of a sequence is ar^{n-1}, where a and r are constants. The first term is 3 and the second term is $-\frac{3}{4}$. Find the values of a and r.

Hence find $\sum_{i=1}^{N} ar^{n-1}$ and $\sum_{i=1}^{\infty} ar^{n-1}$.

4 Evaluate, correct to the nearest whole number,

$$0.99 + 0.99^2 + 0.99^3 + \cdots + 0.99^{99}.$$

5 Find the sum of the infinite series $\frac{1}{10^3} + \frac{1}{10^6} + \frac{1}{10^9} + \ldots$, expressing your answer as a fraction in its lowest terms.

Hence express the infinite recurring decimal 0.108 108 108... as a fraction in its lowest terms.

6 A geometric series has first term 1 and common ratio r. Given that the sum to infinity of the series is 5, find the value of r.

Use a calculator to find, by trial, the least value of n for which the sum of the first n terms of the series exceeds 4.9.

7 In a geometric series, the first term is 12 and the fourth term is $-\frac{3}{2}$. Find the sum, S_n, of the first n terms of the series.

Find the sum to infinity, S_∞, of the series. Find also, by trial, the least value of n for which the magnitude of the difference between S_n and S_∞ is less than 0.001.

8 A geometric series has non-zero first term a and common ratio r, where $0 < r < 1$. Given that $\sum_{i=1}^{8} ar^{n-1} = \frac{1}{2}\sum_{i=1}^{\infty} ar^{n-1}$, find the value of r, correct to 3 decimal places. Given also that the 17th term of the series is 10, find a.

9 A post is being driven into the ground by a mechanical hammer. The distance it is driven by the first blow is 8cm. Subsequently, the distance it is driven by each blow is $\frac{9}{10}$ of the distance it was driven by the previous blow. The post is to be driven a total distance of 70 cm into the ground. Show that at least 20 blows will be needed.

Explain why the post can never be driven a total distance of more than 80 cm into the ground.

10 At the beginning of 1990, an investor decided to invest £6000, believing that the value of the investment should increase, on average, by 6% each year. Show that, if this percentage rate of increase was in fact maintained for 10 years, the value of the investment will be about £10 745.

The investor added a further £6000 at the beginning of each year between 1991 and 1995 inclusive. Assuming that the 6% annual rate of increase continues to apply, show that the total value, in pounds, of the investment at the beginning of the year 2000 may be written as $6000(1.06^5 + 1.06^6 + \cdots + 1.06^{10})$ and evaluate this, correct to the nearest pound.

11 A person wants to borrow £100 000 to buy a house. He intends to pay back a fixed sum of £C at the end of each year, so that after 25 years he has completely paid off the debt. Assuming a steady interest rate of 4% per year, explain why

$$100\,000 = C\left(\frac{1}{1.04} + \frac{1}{1.04^2} + \frac{1}{1.04^3} + \cdots + \frac{1}{1.04^{25}}\right).$$

Calculate the value of C.

12 A person wants to buy a pension which will provide her with an income of £10 000 at the end of each of the next n years. Show that, with a steady interest rate of 5% per year, the pension should cost her

$$£10\,000\left(\frac{1}{1.05}+\frac{1}{1.05^2}+\frac{1}{1.05^3}+\cdots+\frac{1}{1.05^n}\right).$$

Find a simple formula for calculating this sum, and find its value when $n = 10, 20, 30, 40, 50$.

13 The sum of the infinite geometric series $1 + r + r^2 + \ldots$ is k times the sum of the series $1 - r + r^2 - \ldots$, where $k > 0$. Express r in terms of k.

14* A geometric series G has positive first term a, common ratio r and sum to infinity S. The sum to infinity of the even-numbered terms of G (the second, fourth, sixth, ... terms) is $-\frac{1}{2}S$. Find the value of r.

Given that the third term of G is 2, show that the sum to infinity of the odd-numbered terms of G (the first, third, fifth, ... terms) is $\frac{81}{4}$.

15* An infinite geometric series has first term a and sum to infinity b, where $b \neq 0$. Prove that a lies between 0 and $2b$.

16* Find the sum of the geometric series

$$(1-x)+(x^3-x^4)+(x^6-x^7)+\cdots+(x^{3n}-x^{3n+1}).$$

Hence show that the sum of the infinite series $1 - x + x^3 - x^4 + x^6 - x^7 + \ldots$ is equal to $\dfrac{1}{1+x+x^2}$, and state the values of x for which this is valid.

Use a similar method to find the sum of the infinite series $1 - x + x^5 - x^6 + x^{10} - x^{11} + \ldots$.

17 Find the sums of the infinite geometric series

(a) $\sin^2 x° + \sin^4 x° + \sin^6 x° + \sin^8 x° + \ldots$,

(b) $1 - \tan^2 x° + \tan^4 x° - \tan^6 x° + \tan^8 x° - \ldots$,

giving your answers in as simple a form as possible. For what values of x are your results valid?

18* Use the formula to sum the geometric series $1 + (1+x) + (1+x)^2 + \cdots + (1+x)^6$ when $x \neq 0$. By considering the coefficients of x^2, deduce that

$$\binom{2}{2}+\binom{3}{2}+\binom{4}{2}+\binom{5}{2}+\binom{6}{2}=\binom{7}{3}.$$

Illustrate this result on a Pascal triangle.

Write down and prove a general result about binomial coefficients, of which this is a special case.

19* Make tables of values of $1+x$, $1+x+x^2$, $1+x+x^2+x^3$, $1+x+x^2+x^3+x^4$ and $\dfrac{1}{1-x}$ and use them to draw graphs of these functions of x for $-1.5 \leqslant x \leqslant 1.5$.

What do your graphs suggest about the possibility of using the polynomial $1 + x + x^2 + x^3 + \cdots + x^n$ as an approximation to the function $\dfrac{1}{1-x}$?

1 $6\frac{1}{4}$

3 $a = 3, r = -\frac{1}{4}; 2.4\left(1 - \left(-\frac{1}{4}\right)^N\right), 2.4$

4 62

5 $\frac{1}{999}; \frac{4}{37}$

6 $\frac{4}{5}$; 18

7 $8\left(1 - \left(-\frac{1}{2}\right)^n\right)$; 8, 13

8 $r = 0.917; a = 40$

9 The sum of the infinite series is only 80 cm.

10 £56 007

11 6401

12 $£200\,000\left(1 - \frac{1}{1.05^n}\right)$; £77 217, £124 622, £153 725, £171 591, £182 559

13 $r = \dfrac{k-1}{k+1}$

14 $r = -\frac{1}{3}$

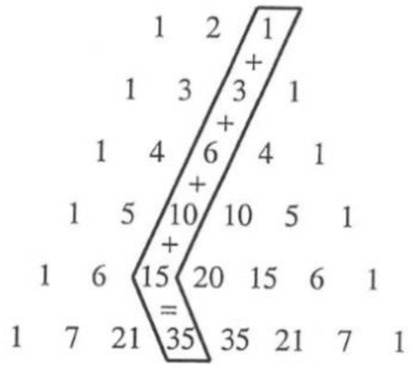
16 $\dfrac{(1-x)(1-x^{3n+3})}{1-x^3}; -1 < x < 1;$

$\dfrac{1}{1 + x + x^2 + x^3 + x^4}, -1 < x < 1$

17 (a) $\tan^2 x^\circ$, $x \neq 90(2n+1)$, where n is an integer

(b) $\cos^2 x^\circ$, $(180n - 45) < x < (180n + 45)$, where n is an integer

18 $\dfrac{(1+x)^7 - 1}{x}$



$$\binom{r}{r} + \binom{r+1}{r} + \binom{r+2}{r} + \cdots + \binom{n}{r} = \binom{n+1}{r+1}$$

19 Possible if $-1 < x < 1$; within these bounds, the larger the value of n the better the approximation.