Sequences (answers at the end)

1 The sequence $u_1, u_2, u_3, \ldots$ is defined by

$$u_1 = 0, \quad u_{r+1} = (2 + u_r)^2.$$

Find the value of u_4.

2 A sequence is defined recursively by $u_{r+1} = 3u_r - 1$ and $u_0 = c$.

(a) Find the first five terms of the sequence if (i) $c = 1$, (ii) $c = 2$, (iii) $c = 0$, (iv) $c = \frac{1}{2}$.

(b) Show that, for each of the values of c in part (a), the terms of the sequence are given by the formula $u_r = \frac{1}{2} + b \times 3^r$ for some value of b.

(c) Show that, if $u_r = \frac{1}{2} + b \times 3^r$ for some value of r, then $u_{r+1} = \frac{1}{2} + b \times 3^{r+1}$.

3 The sequence $u_1, u_2, u_3, \ldots$, where u_1 is a given real number, is defined by

$$u_{n+1} = \sqrt{(4 - u_n)^2}.$$

(a) Given that $u_1 = 1$, evaluate u_2, u_3 and u_4, and describe the behaviour of the sequence.

(b) Given alternatively that $u_1 = 6$, describe the behaviour of the sequence.

(c) For what value of u_1 will all the terms of the sequence be equal to each other? (OCR, adapted)

4 The sequence $u_1, u_2, u_3, \ldots$, where u_1 is a given real number, is defined by $u_{n+1} = u_n^2 - 1$.

(a) Describe the behaviour of the sequence for each of the cases $u_1 = 0$, $u_1 = 1$ and $u_1 = 2$.

(b) Given that $u_2 = u_1$, find exactly the two possible values of u_1.

(c) Given that $u_3 = u_1$, show that $u_1^4 - 2u_1^2 - u_1 = 0$. (OCR)

5 The rth term of an arithmetic progression is $1 + 4r$. Find, in terms of n, the sum of the first n terms of the progression. (OCR)

6 The sum of the first two terms of an arithmetic progression is 18 and the sum of the first four terms is 52. Find the sum of the first eight terms. (OCR)

7 The sum of the first twenty terms of an arithmetic progression is 50, and the sum of the next twenty terms is -50. Find the sum of the first hundred terms of the progression. (OCR)

8 An arithmetic progression has first term a and common difference -1. The sum of the first n terms is equal to the sum of the first $3n$ terms. Express a in terms of n. (OCR)

9 Find the sum of the arithmetic progression 1, 4, 7, 10, 13, 16, ..., 1000.

Every third term of the above progression is removed, i.e. 7, 16, etc. Find the sum of the remaining terms. (OCR)

10 The sum of the first hundred terms of an arithmetic progression with first term a and common difference d is T. The sum of the first 50 odd-numbered terms, i.e. the first, third, fifth, ..., ninety-ninth, is $\frac{1}{2}T - 1000$. Find the value of d. (OCR)

11 In the sequence 1.0, 1.1, 1.2, ..., 99.9, 100.0, each number after the first is 0.1 greater than the preceding number. Find

(a) how many numbers there are in the sequence,

(b) the sum of all the numbers in the sequence. (OCR)

12 The sequence $u_1, u_2, u_3, \ldots$ is defined by $u_n = 2n^2$.

(a) Write down the value of u_3.

(b) Express $u_{n+1} - u_n$ in terms of n, simplifying your answer.

(c) The differences between successive terms of the sequence form an arithmetic progression. For this arithmetic progression, state its first term and its common difference, and find the sum of its first 1000 terms. (OCR)

13 Find formulae for the sums of the following arithmetic progressions.

(a) $\sum_{r=1}^{n} (2 + (r-1)3)$ (b) $\sum_{r=1}^{n} (3 + 2r)$ (c) $\sum_{r=1}^{n} (4 - r)$

14 A small company producing children's toys plans an increase in output. The number of toys produced is to be increased by 8 each week until the weekly number produced reaches 1000. In week 1, the number to be produced is 280; in week 2, the number is 288; etc. Show that the weekly number produced will be 1000 in week 91.

From week 91 onwards, the number produced each week is to remain at 1000. Find the total number of toys to be produced over the first 104 weeks of the plan. (OCR)

15 In 1971 a newly-built flat was sold with a 999-year lease. The terms of the sale included a requirement to pay 'ground rent' yearly. The ground rent was set at £28 per year for the first 21 years of the lease, increasing by £14 to £42 per year for the next 21 years, and then increasing again by £14 at the end of each subsequent period of 21 years.

(a) Find how many complete 21-year periods there would be if the lease ran for the full 999 years, and how many years there would be left over.

(b) Find the total amount of ground rent that would be paid in all of the complete 21-year periods of the lease. (OCR)

16 An arithmetic progression has first term a and common difference 10. The sum of the first n terms of the progression is 10 000. Express a in terms of n, and show that the nth term of the progression is

$$\frac{10\,000}{n} + 5(n-1).$$

Given that the nth term is less than 500, show that $n^2 - 101n + 2000 < 0$ and hence find the largest possible value of n. (OCR)

17 Three sequences are defined recursively by

(a) $u_0 = 0$ and $u_{r+1} = u_r + (2r+1)$,

(b) $u_0 = 0$, $u_1 = 1$ and $u_{r+1} = 2u_r - u_{r-1}$ for $r \geqslant 1$,

(c) $u_0 = 1$, $u_1 = 2$ and $u_{r+1} = 3u_r - 2u_{r-1}$ for $r \geqslant 1$.

For each sequence calculate the first few terms, and suggest a formula for u_r. Check that the formula you have suggested does in fact satisfy all parts of the definition.

18* A sequence F_n is constructed from terms of Pascal sequences as follows:

$$F_0 = \binom{0}{0},\ F_1 = \binom{1}{0} + \binom{0}{1},\ F_2 = \binom{2}{0} + \binom{1}{1} + \binom{0}{2},$$

$$\text{and in general } F_n = \sum_{r=0}^{n} \binom{n-r}{r}.$$

Show that terms of the sequence F_n can be calculated by adding up numbers in Fig. 2.5 (see page 248) along diagonal lines. Verify by calculation that, for small values of n, $F_{n+1} = F_n + F_{n-1}$. (This is called the *Fibonacci sequence*, after the man who introduced algebra from the Arabic world to Italy in about the year 1200.)

Use the Pascal sequence property $\binom{n}{r} + \binom{n}{r+1} = \binom{n+1}{r+1}$ (see Exercise 2B Question 8) to explain why $F_3 + F_4 = F_5$ and $F_4 + F_5 = F_6$.

1 1444

2 (a) (i) 1, 2, 5, 14, 41 (ii) 2, 5, 14, 41, 122
(iii) 0, −1, −4, −13, −40
(iv) $\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}$
(b) (i) $b = \frac{1}{2}$ (ii) $b = \frac{3}{2}$
(iii) $b = -\frac{1}{2}$ (iv) $b = 0$

3 (a) 3, 1, 3; alternately 1 and 3
(b) All terms after the first are 2 (c) 2

4 (a) Alternately 0 and −1; 1, then alternately 0 and −1; gets increasingly large
(b) $\frac{1}{2}(1 \pm \sqrt{5})$

5 $n(2n+3)$

6 168

7 −750

8 $2n-\frac{1}{2}$

9 167 167; 111 445

10 40

11 (a) 991 (b) 50 045.5

12 (a) 18 (b) $2(2n+1)$
(c) $a = 6$, $d = 4$, sum = 2 004 000

13 (a) $\frac{1}{2}n(3n+1)$ (b) $n(n+4)$ (c) $\frac{1}{2}n(7-n)$

14 71 240

15 (a) 47, 12 years left over
(b) £345 450

16 $a = \dfrac{10\,000}{n} - 5(n-1)$; 73

17 (a) 0, 1, 4, 9, 16; r^2 (b) 0, 1, 2, 3, 4; r
(c) 1, 2, 4, 8, 16; 2^r