Applications of differentiation (answers at the end)

1 A particle moves along the x-axis. Its displacement at time t is $x = 6t - t^2$.

(a) What does $\frac{dx}{dt}$ represent?

(b) Is x increasing or decreasing when (i) $t = 1$, (ii) $t = 4$?

(c) Find the greatest (positive) displacement of the particle. How is this connected to your answer to part (a)?

2 The rate at which a radioactive mass decays is known to be proportional to the mass remaining at that time. If, at time t, the mass remaining is m, this means that m and t satisfy the equation

$$\frac{dm}{dt} = -km$$

where k is a positive constant. (The negative sign ensures that $\frac{dm}{dt}$ is negative, which indicates that m is decreasing.)

Write down similar equations which represent the following statements.

(a) The rate of growth of a population of bacteria is proportional to the number, n, of bacteria present.

(b) When a bowl of hot soup is put in the freezer, the rate at which its temperature, θ°C, decreases as it cools is proportional to its current temperature.

3 The rate at which Nasreen's coffee cools is proportional to the difference between its temperature, $\theta°$, and room temperature, $\alpha°$. Sketch a graph of θ against t given that $\alpha = 20$ and that $\theta = 95$ when $t = 0$. State the signs of θ, $\frac{d\theta}{dt}$ and $\frac{d^2\theta}{dt^2}$ for $t > 0$.

4 A car accelerates to overtake a truck. Its initial speed is u, and in a time t after it starts to accelerate it covers a distance x, where $x = ut + kt^2$ and k is a constant.

Use differentiation to show that its speed is then $u + 2kt$, and show that its acceleration is constant.

5 A car is travelling at $20\,\text{m s}^{-1}$ when the driver applies the brakes. At a time t seconds later the car has travelled a further distance x metres, where $x = 20t - 2t^2$. Use differentiation to find expressions for the speed and the acceleration of the car at this time. For how long do these formulae apply?

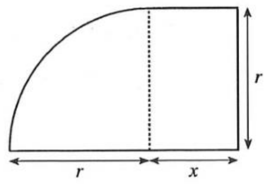
6 Find the least possible value of $x^2 + y^2$ given that $x + y = 10$.

7 The sum of the two shorter sides of a right-angled triangle is 18 cm. Calculate

(a) the least possible length of the hypotenuse,

(b) the greatest possible area of the triangle.

8 The cross-section of an object has the shape of a quarter-circle of radius r adjoining a rectangle of width x and height r, as shown in the diagram.



(a) The perimeter and area of the cross-section are P and A respectively. Express each of P and A in terms of r and x, and hence show that $A = \frac{1}{2}Pr - r^2$.

(b) Taking the perimeter P of the cross-section as fixed, find x in terms of r for the case when the area A of the cross-section is a maximum, and show that, for this value of x, A is a maximum and not a minimum. (OCR)

9 The costs of a firm which makes climbing boots are of two kinds:
Fixed costs (plant, rates, office expenses): £2000 per week;
Production costs (materials, labour): £20 for each pair of boots made.

Market research suggests that, if they price the boots at £30 a pair they will sell 500 pairs a week, but that at £55 a pair they will sell none at all; and between these values the graph of sales against price is a straight line.

If they price boots at £x a pair ($30 \leqslant x \leqslant 55$) find expressions for

(a) the weekly sales,

(b) the weekly receipts,

(c) the weekly costs (assuming that just enough boots are made).

Hence show that the weekly profit, £P, is given by

$$P = -20x^2 + 1500x - 24\,000.$$

Find the price at which the boots should be sold to maximise the profit. (OCR)

10* The manager of a supermarket usually adds a mark-up of 20% to the wholesale prices of all the goods he sells. He reckons that he has a loyal core of F customers and that, if he lowers his mark-up to x% he will attract an extra $k(20 - x)$ customers from his rivals. Each week the average shopper buys goods whose wholesale value is £A. Show that with a mark-up of x% the supermarket will have an anticipated weekly profit of

$$£\tfrac{1}{100}Ax((F + 20k) - kx).$$

Show that the manager can increase his profit by reducing his mark-up below 20% provided that $20k > F$. (OCR)

1 (a) Velocity
(b) (i) increasing (ii) decreasing
(c) 9, occurs when velocity is zero and direction of motion changes

2 (a) $\dfrac{dn}{dt} = kn$ (b) $\dfrac{d\theta}{dt} = -k\theta$

3 +, −, +

5 $(20 - 4t)$ m s^{-1}, -4 m s^{-2}; for $0 \leqslant t \leqslant 5$

6 50

7 (a) $9\sqrt{2}$ cm (b) $40\frac{1}{2}$ cm^2

8 (a) $P = 2x + 2r + \frac{1}{2}\pi r$, $A = \frac{1}{4}\pi r^2 + rx$
(b) $x = \frac{1}{4}r(4 - \pi)$

9 (a) $1100 - 20x$
(b) £$x(1100 - 20x)$
(c) £$(24\,000 - 400x)$
£37.50