Trigonometry (answers at the end)

1 Write down the period of each of the following.

(a) $\sin x$ (b) $\tan 2x$ (OCR)

2 By considering the graph of $y = \cos x$, or otherwise, express the following in terms of $\cos x$.

(a) $\cos(360^\circ - x)$ (b) $\cos(x + 180^\circ)$ (OCR)

3 Draw the graph of $y = \cos \frac{1}{2}\theta$ for θ in the interval $-360^\circ \leqslant \theta \leqslant 360^\circ$. Mark clearly the coordinates of the points where the graph crosses the θ and y-axes.

4 Solve the following equations for θ, giving your answers in the interval $0^\circ \leqslant \theta \leqslant 360^\circ$.

(a) $\tan\theta = 0.4$ (b) $\sin 2\theta = 0.4$ (OCR)

5 Solve the equation $3\cos 2x = 2$, giving all the solutions in the interval $0^\circ \leqslant x \leqslant 180^\circ$ correct to the nearest 0.1. (OCR)

6 (a) Give an example of a trigonometric function which has a period of 180°.

(b) Solve for x the equation $\sin 3x = 0.5$, giving all solutions in the interval $0^\circ < x < 180^\circ$. (OCR)

7 Find all values of θ, $0^\circ \leqslant \theta \leqslant 360^\circ$, for which $2\cos(\theta + 30^\circ) = 1$. (OCR)

8 (a) Express $\sin 2x + \cos(90^\circ - 2x)$ in terms of a single trigonometric function.

(b) Hence, or otherwise, find all values of x in the interval $0^\circ \leqslant x \leqslant 360^\circ$ for which $\sin 2x + \cos(90^\circ - 2x) = -1$. (OCR)

9 Find the least positive value of the angle A for which

(a) $\sin A = 0.2$ and $\cos A$ is negative, (b) $\tan A = -0.5$ and $\sin A$ is negative,

(c) $\cos A = \sin A$ and both are negative, (d) $\sin A = -0.2275$ and $A > 360^\circ$.

10 Prove the following identities.

(a) $\dfrac{1}{\sin\theta} - \sin\theta \equiv \dfrac{\cos\theta}{\tan\theta}$ (b) $\dfrac{1-\sin\theta}{\cos\theta} \equiv \dfrac{\cos\theta}{1+\sin\theta}$

(c) $\dfrac{1}{\tan\theta} + \tan\theta \equiv \dfrac{1}{\sin\theta\cos\theta}$ (d) $\dfrac{1-2\sin^2\theta}{\cos\theta+\sin\theta} \equiv \cos\theta - \sin\theta$

11 For each of the following functions, determine the maximum and minimum values of y and the least positive values of x at which these occur.

(a) $y = 1 + \cos 2x$ (b) $y = 5 - 4\sin(x + 30^\circ)$

(c) $y = 29 - 20\sin(3x - 45^\circ)$ (d) $y = 8 - 3\cos^2 x$

(e) $y = \dfrac{12}{3+\cos x}$ (f) $y = \dfrac{60}{1+\sin^2(2x - 15^\circ)}$

12 Solve the following equations for θ, giving solutions in the interval $0^\circ \leqslant \theta \leqslant 360^\circ$.

(a) $\sin\theta = \tan\theta$ (b) $2 - 2\cos^2\theta = \sin\theta$

(c) $\tan^2\theta - 2\tan\theta = 1$ (d) $\sin 2\theta - \sqrt{3}\cos 2\theta = 0$

13 The function t is defined by $t(x) = \tan 3x$.

(a) State the period of $t(x)$.

(b) Solve the equation $t(x) = \frac{1}{2}$ for $0° \leqslant x \leqslant 180°$.

(c) Deduce the smallest positive solution of each of the following equations.

(i) $t(x) = -\frac{1}{2}$

(ii) $t(x) = 2$ (OCR)

14 In each of the following, construct a formula involving a trigonometric function which could be used to model the situations described.

(a) Water depths in a canal vary between a minimum of 3.6 metres and a maximum of 6 metres over 24-hour periods.

(b) Petroleum refining at a chemical plant is run on a 10-day cycle, with a minimum production of 15 000 barrels per day and a maximum of 28 000 barrels per day.

(c) At a certain town just south of the Arctic circle, the number of hours of daylight varies between 2 and 22 hours during a 360-day year.

15 A tuning fork is vibrating. The displacement, y centimetres, of the tip of one of the prongs from its rest position after t seconds is given by

$$y = 0.1 \sin(100\,000t).$$

Find

(a) the greatest displacement and the first time at which it occurs,

(b) the time taken for one complete oscillation of the prong,

(c) the number of complete oscillations per second of the tip of the prong,

(d) the total time during the first complete oscillation for which the tip of the prong is more than 0.06 centimetres from its rest position.

16 One end of a piece of elastic is attached to a point at the top of a door frame and the other end hangs freely. A small ball is attached to the free end of the elastic. When the ball is hanging freely it is pulled down a small distance and then released, so that the ball oscillates up and down on the elastic. The depth d centimetres of the ball from the top of the door frame after t seconds is given by

$$d = 100 + 10 \cos 500t.$$

Find

(a) the greatest and least depths of the ball,

(b) the time at which the ball first reaches its highest position,

(c) the time taken for a complete oscillation,

(d) the proportion of the time during a complete oscillation for which the depth of the ball is less than 99 centimetres.

17* An oscillating particle has displacement y metres, where y is given by $y = a\sin(kt + \alpha)$, where a is measured in metres, t is measured in seconds and k and α are constants. The time for a complete oscillation is T seconds. Find

(a) k in terms of T,

(b) the number, in terms of k, of complete oscillations per second.

18* The population, P, of a certain type of bird on a remote island varies during the course of a year according to feeding, breeding, migratory seasons and predator interactions. An ornithologist doing research into bird numbers for this species attempts to model the population on the island with the annually periodic equation

$$P = N - C\cos\omega t,$$

where N, C and ω are constants, and t is the time in weeks, with $t = 0$ representing midnight on the first of January.

(a) Taking the period of this function to be 50 weeks, find the value of ω.

(b) Use the equation to describe, in terms of N and C,

(i) the number of birds of this species on the island at the start of each year;

(ii) the maximum number of these birds, and the time of year when this occurs.

19* The road to an island close to the shore is sometimes covered by the tide. When the water rises to the level of the road, the road is closed. On a particular day, the water at high tide is a height 4.6 metres above mean sea level. The height, h metres, of the tide is modelled by using the equation $h = 4.6\cos kt$, where t is the time in hours from high tide; it is also assumed that high tides occur every 12 hours.

(a) Determine the value of k.

(b) On the same day, a notice says that the road will be closed for 3 hours. Assuming that this notice is correct, find the height of the road above sea level, giving your answer correct to two decimal places.

(c) In fact, a road repair has raised its level, and it is impassable for only 2 hours 40 minutes. By how many centimetres has the road level been raised? (OCR)

20* A simple model of the tides in a harbour on the south coast of Cornwall assumes that they are caused by the attractions of the Sun and the Moon. The magnitude of the attraction of the Moon is assumed to be nine times the magnitude of the attraction of the Sun. The period of the Sun's effect is taken to be 360 days and that of the Moon is 30 days. A model for the height, h metres, of the tide (relative to a mark fixed on the harbour wall), at t days, is

$$h = A\cos\alpha t + B\cos\beta t,$$

where the term $A\cos\alpha t$ is the effect due to the Sun, and the term $B\cos\beta t$ is the effect due to the Moon. Given that $h = 5$ when $t = 0$, determine the values of A, B, α and β. (OCR, adapted)

1 (a) $360°$ (b) $90°$

2 (a) $\cos x°$ (b) $-\cos x°$

3 $(0°,1)$, $(\pm 180°, 0)$

4 (a) $21.8°$, $201.8°$
(b) $11.8°$, $78.2°$, $191.8°$, $258.2°$

5 $24.1°$, $155.9°$

6 (a) Examples are $\tan x$, $\sin 2x$, $\cos 2x$
(b) $10°$, $50°$, $130°$, $170°$

7 $30°$, $270°$

8 (a) $2 \sin 2x$ (b) $105°$, $165°$, $285°$, $345°$

9 (a) $168.5°$ (b) $333.4°$ (c) $225°$ (d) $553.1°$

11 (a) 2, 0; $180°$, $90°$ (b) 9, 1; $240°$, $60°$
(c) 49, 9; $105°$, $45°$ (d) 8, 5; $90°$, $180°$
(e) 6, 3; $180°$, $360°$ (f) 60, 30; $7\frac{1}{2}°$, $52\frac{1}{2}°$

12 (a) $0°$, $180°$, $360°$ (b) $0°$, $30°$, $150°$, $180°$, $360°$
(c) $67\frac{1}{2}°$, $157\frac{1}{2}°$, $247\frac{1}{2}°$, $337\frac{1}{2}°$
(d) $30°$, $120°$, $210°$, $300°$

13 (a) $60°$ (b) $8.9°$, $68.9°$, $128.9°$
(c) (i) $51.1°$ (ii) $21.1°$

14 (a) $4.8 \pm 1.2 \sin 15t$ or $4.8 \pm 1.2 \cos 15t$
(b) $21\,500 \pm 6500 \sin 36t$ or
$21\,500 \pm 6500 \cos 36t$
(c) $12 \pm 10 \sin t$ or $12 \pm 10 \cos t$

15 (a) 0.1 cm, 0.0009 seconds
(b) 0.0036 seconds
(c) 278 (d) 0.002 13 seconds

16 (a) 110 cm and 90 cm (b) 0.36 seconds
(c) 0.72 seconds (d) 0.468

17 (a) $k = \dfrac{360}{T}$ (b) $\dfrac{k}{360}$

18 (a) 7.2
(b) (i) $N - C$ (ii) $N + C$, after 25 weeks

19 (a) 30 (b) 3.25 m (c) 27

20 $A = 0.5$, $B = 4.5$, $\alpha = 1$, $\beta = 12$